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Perturbation analysis of the maximal solution of the matrix equation $X + A^*X^{-1}A = P$

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Abstract

Consider the nonlinear matrix equation

$$X + A^*X^{-1}A = P,$$

where A, P are $n \times n$ complex matrices with P Hermitian positive definite, and A^* denotes the conjugate transpose of a matrix A. In this paper a perturbation bound for the maximal solution of the matrix equation and a computable error bound for approximate solutions are derived by using an elegant property of the maximal solution. The results are illustrated by using numerical examples. © 2001 Elsevier Science Inc. All rights reserved.

Keywords: Nonlinear matrix equation; Maximal solution; Perturbation bound; Condition number

1. Introduction

Consider the nonlinear matrix equation

$$X + A^*X^{-1}A = P, \tag{1.1}$$

where $A, P \in \mathbb{C}^{n\times n}$ with P Hermitian positive definite. Here $\mathbb{C}^{n\times n}$ denotes the set of all $n \times n$ complex matrices, and A^* the conjugate transpose of a matrix A. It is often required to find the Hermitian positive definite solution X of the matrix equation (1.1).

This kind of equation arises in various areas of applications, including control theory, ladder networks, dynamic programming, stochastic filtering, statistics, and so on. For further details see [1–3] and the references contained therein.

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By means of the shorted operator approach Anderson et al. [1] gave a necessary and sufficient condition for the existence of a Hermitian positive definite solution of the matrix equation (1.1) and it was noted there that the solution is not unique. Recently some other necessary and sufficient conditions were given in [3,6]. It was proved in [3] that if the matrix equation (1.1) has a Hermitian positive definite solution, then all its Hermitian solutions are positive definite; and moreover, it has a maximal solution X_L and a minimal solution X_S in the sense that $X_S \leqslant X \leqslant X_L$ for any Hermitian positive definite solution X. Here $X \geqslant Y$ means that $X - Y$ is positive semidefinite. From the application point of view, the maximal solution is more important.

It is worthwhile to point out that although several approaches have been developed for finding the maximal solution of the matrix equation (1.1) (see [2,5,7], and references contained therein), relatively little attention has been paid to the perturbation analysis of this problem.

The central question of perturbation theory for the matrix equation (1.1) should be as follows: How does the maximal solution X_L change when the coefficient matrices A and P are subject to perturbations? In view of [3, Section 6], the theory of invariant Lagrangian subspaces can be applied to this problem. The theory is completely analogous to the one for the discrete Riccati equation. In this perspective, it follows from [4] that X_L is stable under small perturbations of A and P. However, from this we cannot get explicit perturbation bounds, which are important.

It was proved in [5] that the maximal solution is well conditioned. In this paper, by using this property we obtain a perturbation bound for the maximal solution of the matrix equation and a computable error bound for approximate solutions (Section 3). The results are illustrated by using numerical examples (Section 4).

Throughout this paper we use X_L to denote the maximal solution of the matrix equation (1.1). I denotes the identity matrix of order n and $\|\cdot\|$ denotes the matrix 2-norm. A^* represents the conjugate transpose of a matrix A.

2. Preliminary lemmas

In this section we first give some preliminary results which play a fundamental role in this paper. By [2, Theorem 13] and [3, Theorem 3.4] we immediately obtain the following lemma.

Lemma 2.1. *If $\|A\| \leqslant \frac{1}{2}$, then the matrix equation $X + A^*X^{-1}A = I$ has a maximal solution X_L.*

Remark 2.1. It has been proved in [2] that if A is normal, then the matrix equation $X + A^*X^{-1}A = I$ has a maximal solution if and only if $\|A\| \leqslant \frac{1}{2}$. This shows that we cannot improve the condition of Lemma 2.1 if only a spectral norm bound is used.

The following lemma immediately follows from [5, Theorem 1.1].

Lemma 2.2. *If* $\|A\| < \frac{1}{2}$, *then the maximal solution* X_L *of the matrix equation* $X + A^*X^{-1}A = I$ *satisfies* $\|X_L^{-1}\| < 2$. *Moreover*, $\|X^{-1}\| > 2$ *for any other solution* X (*including non-Hermitian ones*).

Using Lemmas 2.1 and 2.2 we can easily prove the following lemma.

Lemma 2.3. *If* $\|A\| < \frac{1}{2}\|P^{-1}\|^{-1}$, *then the maximal solution* X_L *of the matrix equation* (1.1) *exists and satisfies*

$$\|X_L^{-1}\| < 2\|P^{-1}\|. \tag{2.1}$$

Moreover, *for any other solution* X (*including non-Hermitian ones*) *we have*

$$\|X^{-1}\| > 2\|P\|^{-1}. \tag{2.2}$$

Proof. It is easy to see that the matrix equation (1.1) is equivalent to

$$Y + B^*Y^{-1}B = I, \tag{2.3}$$

where $Y = P^{-1/2}XP^{-1/2}$, $B = P^{-1/2}AP^{-1/2}$, $P^{-1/2} = (P^{1/2})^{-1}$, $P^{1/2}$ denotes the Hermitian positive definite square root of P. It follows from the assumption that:

$$\|B\| \leqslant \|P^{-1/2}\|^2\|A\| = \|P^{-1}\|\,\|A\| < \tfrac{1}{2},$$

and so, by Lemmas 2.1 and 2.2, the maximal solution Y_L of (2.3) exists and satisfies that $\|Y_L^{-1}\| < 2$, and moreover, $\|Y^{-1}\| > 2$ for any other solution Y. Consequently, the maximal solution X_L of the matrix equation (1.1) exists and satisfies that

$$\|X_L^{-1}\| = \|P^{-1/2}Y_L^{-1}P^{-1/2}\| \leqslant \|P^{-1}\|\|Y_L^{-1}\| < \tfrac{1}{2}.$$

Moreover, we have

$$\|X^{-1}\| \geqslant \|P\|^{-1}\|P^{1/2}X^{-1}P^{1/2}\| = \|P\|^{-1}\|Y^{-1}\| > 2\|P\|^{-1}$$

for any other solution X of the matrix equation (1.1). □

Lemma 2.4. *If* $\|A\| < \frac{1}{2}\|P^{-1}\|^{-1}$, *then the maximal solution* X_L *of the matrix equation* (1.1) *satisfies*

$$\tfrac{1}{2}\|P\| \leqslant \|X_L\| \leqslant \|P\|. \tag{2.4}$$

Proof. Noting that

$$\|A\| < \tfrac{1}{2}\|P^{-1}\|^{-1} \leqslant \tfrac{1}{2}\|P\|,$$

by Lemma 2.3, we have

$$\begin{aligned}\|X_L\| &= \|P - A^* X_L^{-1} A\| \\ &\geqslant \|P\| - \|A\|^2 \|X_L^{-1}\| \\ &\geqslant \|P\| - 2\|A\|^2 \|P^{-1}\| \\ &\geqslant \|P\| - \|A\| \\ &\geqslant \tfrac{1}{2}\|P\|.\end{aligned}$$

On the other hand, from

$$X_L = P - A^* X_L^{-1} A,$$

it is easy to see that $X_L \leqslant P$, and so we have $\|X_L\| \leqslant \|P\|$. $\square$

3. Main results

Our main result is the following theorem.

Theorem 3.1. *Let $A,\ \widetilde{A},\ P,\ \widetilde{P} \in \mathbb{C}^{n\times n}$ with P and $\widetilde{P}$ Hermitian positive definite. If*

$$\|A\|\|P^{-1}\| < \tfrac{1}{2}, \tag{3.1}$$

$$\|\widetilde{A} - A\| < \tfrac{1}{2}\left(\tfrac{1}{2} - \|A\|\,\|P^{-1}\|\right)\|P^{-1}\|^{-1}, \tag{3.2}$$

$$\|\widetilde{P} - P\| \leqslant \left(\tfrac{1}{2} - \|A\|\,\|P^{-1}\|\right)\|P^{-1}\|^{-1}, \tag{3.3}$$

then the maximal solutions X_L and $\widetilde{X}_L$ of the matrix equations

$$X + A^* X^{-1} A = P \quad \text{and} \quad \widetilde{X} + \widetilde{A}^* \widetilde{X}^{-1} \widetilde{A} = \widetilde{P} \tag{3.4}$$

exist and satisfy that

$$\frac{\|\widetilde{X}_L - X_L\|}{\|X_L\|} \leqslant \frac{1}{\frac{1}{2} - \|A\|\,\|P^{-1}\|}\left(\frac{\|\widetilde{A} - A\|}{\|A\|} + \frac{\|\widetilde{P} - P\|}{\|P\|}\right). \tag{3.5}$$

Proof. Let $S = \widetilde{P} - P$. It is easy to verify that

$$\widetilde{P}^{-1} = P^{-1} - P^{-1} S \widetilde{P}^{-1}.$$

Taking norm and using (3.3), we get

$$\begin{aligned}\|\widetilde{P}^{-1}\| &\leqslant \|P^{-1}\| + \|P^{-1}\|\,\|S\|\,\|\widetilde{P}^{-1}\|\\ &\leqslant \|P^{-1}\| + \left(\tfrac{1}{2} - \|A\|\,\|P^{-1}\|\right)\|\widetilde{P}^{-1}\|,\end{aligned}$$

and so we have

$$\|\widetilde{P}^{-1}\| \leqslant \frac{\|P^{-1}\|}{\frac{1}{2} + \|A\|\,\|P^{-1}\|}. \tag{3.6}$$

Combining (3.2) and (3.6), we obtain

$$\begin{aligned}\|\widetilde{A}\|\,\|\widetilde{P}^{-1}\| &\leqslant (\|A\| + \|\widetilde{A} - A\|)\|\widetilde{P}^{-1}\|\\ &< \left(\|A\| + \frac{\frac{1}{2} - \|A\|\,\|P^{-1}\|}{2\|P^{-1}\|}\right)\frac{\|P^{-1}\|}{\frac{1}{2} + \|A\|\,\|P^{-1}\|}\\ &= \tfrac{1}{2}.\end{aligned} \tag{3.7}$$

Consequently, by Lemmas 2.3 and 2.4, this, together with (3.1), implies that the maximal solutions X_L and $\widetilde{X}_L$ exist and satisfy that

$$\|X_L^{-1}\| < 2\,\|P^{-1}\|, \quad \|\widetilde{X}_L^{-1}\| < 2\,\|\widetilde{P}^{-1}\|, \tag{3.8}$$

$$\tfrac{1}{2}\|P\| \leqslant \|X_L\| \leqslant \|P\|, \quad \tfrac{1}{2}\|\widetilde{P}\| \leqslant \|\widetilde{X}_L\| \leqslant \|\widetilde{P}\|. \tag{3.9}$$

Moreover, it follows from (3.1) and (3.6) that

$$\|A\|\,\|\widetilde{P}^{-1}\| \leqslant \frac{\|A\|\,\|P^{-1}\|}{\frac{1}{2} + \|A\|\,\|P^{-1}\|} < \tfrac{1}{2}. \tag{3.10}$$

Now let

$$Z = \widetilde{X}_L - X_L, \quad E = \widetilde{A} - A.$$

Note that X_L and $\widetilde{X}_L$ satisfy that

$$X_L + A^* X_L^{-1} A = P \quad \text{and} \quad \widetilde{X}_L + \widetilde{A}^* \widetilde{X}_L^{-1} \widetilde{A} = \widetilde{P}.$$

We get

$$Z - A^* \widetilde{X}_L^{-1} Z X_L^{-1} A + A^* \widetilde{X}_L^{-1} E + E^* \widetilde{X}_L^{-1} \widetilde{A} = S. \tag{3.11}$$

Note that it follows from (3.8) and (3.10) that

$$\begin{aligned}\|Z - A^* \widetilde{X}_L^{-1} Z X_L^{-1} A\| &\geqslant \|Z\| - \|A\|^2\,\|\widetilde{X}_L^{-1}\|\,\|Z\|\,\|X_L^{-1}\|\\ &\geqslant \|Z\|(1 - \|A\|\,\|X_L^{-1}\|)\\ &\geqslant 2\,\|Z\|\left(\tfrac{1}{2} - \|A\|\,\|P^{-1}\|\right)\\ &= 2\,\delta\|Z\|,\end{aligned}$$

where $\delta = \frac{1}{2} - \|A\|\,\|P^{-1}\| > 0$, and hence we have

$$\begin{aligned}2\delta\|Z\| &\leqslant \|Z - A^*\widetilde{X}_L^{-1}ZX_L^{-1}A\| \\ &= \|S - A^*\widetilde{X}_L^{-1}E - E^*\widetilde{X}_L^{-1}\widetilde{A}\| \\ &\leqslant \|S\| + \|A\|\,\|\widetilde{X}_L^{-1}\|\,\|E\| + \|E\|\,\|\widetilde{X}_L^{-1}\|\,\|\widetilde{A}\| \\ &\leqslant \|S\| + 2\|E\|. \end{aligned} \tag{3.12}$$

The last inequality follows from

$$\|A\|\,\|\widetilde{X}_L^{-1}\| \leqslant 2\,\|A\|\,\|\widetilde{P}^{-1}\| < 1 \quad \text{and} \quad \|\widetilde{A}\|\,\|\widetilde{X}_L^{-1}\| \leqslant 2\,\|\widetilde{A}\|\,\|\widetilde{P}^{-1}\| < 1.$$

Noting that

$$\|X_L\| \geqslant \tfrac{1}{2}\|P\| \geqslant \tfrac{1}{2}\|P^{-1}\|^{-1} \geqslant \|A\|,$$

we get

$$\frac{\|P\|}{\|X_L\|} \leqslant 2, \qquad \frac{\|A\|}{\|X_L\|} \leqslant 1. \tag{3.13}$$

Combining (3.12) and (3.13) gives that

$$\begin{aligned}\frac{\|Z\|}{\|X_L\|} &\leqslant \frac{1}{2\delta}\left(\frac{\|S\|}{\|X_L\|} + 2\frac{\|E\|}{\|X_L\|}\right) \\ &= \frac{1}{2\delta}\left(\frac{\|P\|}{\|X_L\|}\frac{\|S\|}{\|P\|} + 2\frac{\|A\|}{\|X_L\|}\frac{\|E\|}{\|A\|}\right) \\ &\leqslant \frac{1}{\delta}\left(\frac{\|S\|}{\|P\|} + \frac{\|E\|}{\|A\|}\right),\end{aligned}$$

which is the desired result. □

By Theorem 3.1 we see that the perturbation property of the maximal solution X_L of the matrix equation (1.1) is closely related to the number

$$\kappa(A, P) = \frac{1}{\frac{1}{2} - \|A\|\,\|P^{-1}\|}. \tag{3.14}$$

When $\kappa(A, P)$ is not too large, the solution is insensitive to perturbation in data A and P; when $\kappa(A, P)$ is relatively large, generally speaking, it is very sensitive. Hence we refer the number $\kappa(A, P)$ as the *condition number* of the matrix equation (1.1) at its maximal solution X_L.

An important special case of the matrix equation (1.1) is $P = I$, that is,

$$X + A^*X^{-1}A = I.$$

Applying Theorem 3.1 to this equation we immediately obtain:

Corollary 3.2. *Let* $A, \widetilde{A} \in \mathbb{C}^{n\times n}$. *If*

$$\|A\| < \tfrac{1}{2}, \qquad \|\widetilde{A} - A\| < \tfrac{1}{2}\left(\tfrac{1}{2} - \|A\|\right),$$

then the maximal solutions X_L *and* $\widetilde{X}_L$ *of the matrix equations*

$$X + A^* X^{-1} A = I \quad \text{and} \quad \widetilde{X} + \widetilde{A}^* \widetilde{X}^{-1} \widetilde{A} = I$$

exist and satisfy that

$$\frac{\|\widetilde{X}_L - X_L\|}{\|X_L\|} \leqslant \frac{1}{\frac{1}{2} - \|A\|} \frac{\|\widetilde{A} - A\|}{\|A\|}.$$

Let $\widetilde{X}$ approximate the maximal solution of the matrix equation (1.1). For example, $\widetilde{X}$ may come from a numerical algorithm for approximating the maximal solution. A practical question is: How to determine the accuracy of $\widetilde{X}$? The following result gives a computable error bound of $\widetilde{X}$.

Theorem 3.3. *Let* $\widetilde{X}$ *approximate the maximal solution* X_L *of the matrix equation* (1.1). *If*

$$\|A\|\,\|P^{-1}\| < \tfrac{1}{2} \quad \text{and} \quad \|\widetilde{X}^{-1}\| \leqslant 2\,\|(\widetilde{X} + A^* \widetilde{X}^{-1} A)^{-1}\|,$$

and if the residual

$$R(\widetilde{X}) \equiv \widetilde{X} + A^* \widetilde{X}^{-1} A - P$$

satisfies that

$$\|R(\widetilde{X})\| \leqslant \left(\tfrac{1}{2} - \|A\|\,\|P^{-1}\|\right) \|P^{-1}\|^{-1},$$

then

$$\frac{\|\widetilde{X} - X_L\|}{\|X_L\|} \leqslant \frac{1}{\frac{1}{2} - \|A\|\,\|P^{-1}\|} \frac{\|R(\widetilde{X})\|}{\|P\|}. \tag{3.15}$$

Proof. Note that $\widetilde{X}$ is a solution of the matrix equation

$$X + A^* X^{-1} A = \widetilde{P},$$

where $\widetilde{P} = P + R(\widetilde{X})$, and satisfies that

$$\|\widetilde{X}^{-1}\| \leqslant 2\,\|(\widetilde{X} + A^* \widetilde{X}^{-1} A)^{-1}\| = 2\,\|\widetilde{P}^{-1}\|.$$

Thus, following the same lines as the proof of (3.5), inequality (3.15) can be proved. $\square$

4. Numerical examples

To illustrate the results of the previous section, in this section several interesting numerical examples are given, which were carried out using MATLAB on a PC Pentium $III/500$ computer, with machine epsilon $\varepsilon = 2.2 \times 10^{-16}$.

Example 4.1. Consider the matrix equation

$$X + A_k^* X^{-1} A_k = I,$$

with $A_k = (\delta_k/\|A\|)A$, where

$$\delta_k = \frac{1}{2} - 10^{-k}, \quad A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

The maximal solution of the equation is given by

$$X_L^{(k)} = \frac{1}{2}(I + (I - 4A_k^* A_k)^{1/2}).$$

See [2,6] for more details.

We now consider perturbation bound for the maximal solution $X_L^{(k)}$ when the coefficient matrix A_k is perturbed to $A_{kj} = A_k + \delta_k^{j+1} A_0$, where

$$A_0 = \frac{1}{\|C^{\mathrm{T}} + C\|}(C^{\mathrm{T}} + C),$$

C is a random matrix generated by MATLAB function **rand**. By Corollary 3.2, we can compute perturbation bound:

$$r_{kj} \equiv \frac{\|X_L^{(k)} - X_L^{(kj)}\|}{\|X_L^{(k)}\|} \leqslant \frac{1}{\frac{1}{2} - \|A_k\|} \frac{\|A_k - A_{kj}\|}{\|A_k\|} \equiv \kappa_{kj},$$

where

$$X_L^{(kj)} = \frac{1}{2}(I + (I - 4A_{kj}^* A_{kj})^{1/2}).$$

Some results are listed in Table 1.

The results listed in Table 1 show that the perturbation bound given by Corollary 3.2 is relatively sharp.

Example 4.2. Consider the matrix equation (1.1) with the coefficient matrices

$$A = \frac{1}{10}\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix}, \quad P = X + A^* X^{-1} A,$$

Table 1

j	1	2	3	4	5
r_{1j}	1.34e − 002	1.30e − 003	1.29e − 004	1.29e − 005	1.29e − 006
κ_{1j}	2.50e − 001	2.50e − 002	2.50e − 003	2.50e − 004	2.50e − 005
r_{2j}	4.75e − 004	4.74e − 006	4.74e − 008	4.74e − 010	4.74e − 012
κ_{2j}	2.04e − 002	2.04e − 004	2.04e − 006	2.04e − 008	2.04e − 010
r_{3j}	1.52e − 005	1.52e − 008	1.52e − 011	1.30e − 014	0.00e + 000
κ_{3j}	2.00e − 003	2.00e − 006	2.00e − 009	2.00e − 012	2.00e − 015

where $X = \mathrm{diag}(1, 2, 3, 2, 1)$, which is just the maximal solution of the associate matrix equation (1.1).

We now consider perturbation bound for the maximal solution X when the coefficient matrices A and P are perturbed to $A_j = A + 10^{-2j} A_0$ and $P_j = P + 10^{-2j} P_0$, respectively, where

$$A_0 = \frac{1}{\|C\|} C, \qquad P_0 = \frac{1}{\|C^{\mathrm{T}} + C\|}(C^{\mathrm{T}} + C),$$

C is a random matrix, which is generated by MATLAB function **rand**. By Theorem 3.1, we can compute perturbation bound

$$r_j \equiv \frac{\|X - \widetilde{X}_j\|}{\|X\|} \leqslant \frac{1}{\frac{1}{2} - \|A\|\,\|P^{-1}\|}\left(\frac{\|A_j - A\|}{\|A\|} + \frac{\|P_j - P\|}{\|P\|}\right) \equiv \kappa_j,$$

where $\widetilde{X}_j$ is computed by applying MATLAB function **dare** (i.e., $\widetilde{X}_j =$ **dare**$(0, I, P_j, 0, A_j^{\mathrm{T}})$), which is a discrete-time algebraic Riccati equation solver. Some results are listed in Table 2.

The results listed in Table 2 show that the perturbation bound given by Theorem 3.1 is relatively sharp.

Example 4.3. Consider the matrix equation (1.1) with the same coefficient matrices A and P as in Example 4.2. Let the approximate solution X_j of X be given, where $X_j = X + 10^{-2j} X_0$ with

$$X_0 = \frac{1}{\|C^{\mathrm{T}} + C\|}(C^{\mathrm{T}} + C),$$

Table 2

j	1	2	3	4	5
r_j	3.56e − 003	3.58e − 005	3.58e − 007	3.58e − 009	3.58e − 011
κ_j	1.83e − 001	1.83e − 003	1.83e − 005	1.83e − 007	1.83e − 009

Table 3

j	1	2	3	4	5
r_j	3.33e − 003	3.33e − 005	3.33e − 007	3.33e − 009	3.33e − 011
κ_j	1.73e − 002	1.73e − 004	1.73e − 006	1.73e − 008	1.73e − 010

C is a random matrix, which is also generated by **rand**. Assume that the solution X is unknown. By Theorem 3.3, we can compute error bound for X_j:

$$r_j \equiv \frac{\|X_j - X\|}{\|X\|} \leqslant \frac{1}{\frac{1}{2} - \|A\|\,\|P^{-1}\|} \frac{\|R(X_j)\|}{\|P\|} \equiv \kappa_j,$$

where $R(X_j) = X_j + A^* X_j^{-1} A - P$. Some results are listed in Table 3.

The results listed in Table 3 show that the error bound given by Theorem 3.3 is fairly sharp.

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